SLOPE: A NETWORK OF CONNECTED COMPONENTS

Courtney R. Nagle
Penn State Erie, The Behrend College
crt12@psu.edu

Deborah Moore-Russo
University at Buffalo
dam29@buffalo.edu

In this study, we build on previous work on conceptualizations of slope to suggest a framework of slope across the landscape of the mathematics curriculum. Data from multiple studies on students', teachers', and college instructors' conceptualizations of slope are revisited in light of theory on procedural versus conceptual understanding and visual versus analytic interpretations of slope. This synthesis leads to a description of five key slope components. A detailed description of each component is provided and implications for future research are discussed.

Keywords: Algebra and Algebraic Thinking; High School Education; Middle School Education

Introduction

Slope is a key mathematical concept revisited throughout the mathematics curriculum. In addition to being an important mathematical concept of its own right, slope is an important prerequisite concept for advanced mathematical thinking, extending to include the notion of rate of change in precalculus and the concept of derivative in calculus (Carlson, Oehrtman, & Engelke, 2010; Confrey & Smith, 1995; Noble, Nemirovsky, Wright, & Tierney, 2001). Various studies suggest students struggle to understand slope (Barr, 1981; Carlson, Jacobs, Coe, Larsen, & Hsu, 2002; Lobato & Thanheiser, 2002; Orton, 1984; Stump, 2001b; Teuscher & Reys, 2010; Thompson, 1994). In particular, research shows that students' knowledge of slope does not transfer between problem types (e.g., qualitative versus quantitative and mathematical versus real world settings) and that students do not relate the concepts of slope and rate of change (Hattikudur et. al., 2011; Stump, 2001b; Lobato & Siebert, 2002; Lobato & Thanheiser, 2002; Teuscher & Reys, 2010). Students' difficulties connecting notions of slope are understandable in light of the variety of conceptualizations emphasized in the mathematics curriculum. Moore-Russo and colleagues (Moore-Russo, Conner, & Rugg, 2011; Mudaly & Moore-Russo, 2011; Stanton & Moore-Russo, 2012) have suggested eleven conceptualizations of slope, outlined in Table 1, based on their own research and the earlier work of Sheryl Stump (1999, 2001a, 2001b).

Table 1: Conceptualizations of Slope

Category	Slope as …
Geometric Ratio (G)	Rise over run of a graph of a line; ratio of vertical displacement to horizontal displacement of a line's graph
Algebraic Ratio (A)	Change in y over change in x; ratio with algebraic expressions (often seen as either $\Delta y/\Delta x$ or $(y_2 - y_1)/(x_2 - x_1)$)
Physical Property (P)	Property of line often described using expressions like grade, incline, pitch, steepness, slant, tilt, and "how high a line goes up"
Functional Property (F)	(Constant) rate of change between variables; sometimes seen in responses involving related rates
Parametric Coefficient (PC)	The variable m (or its numeric value) found in y=mx+b and $(y_2 - y_1)=m(x_2 - x_1)$
Trigonometric	Property related to the angle a line makes with a horizontal line; tangent of a

Conception (T)	line's angle of inclination/decline; direction component of a vector
Calculus Conception (C)	Limit; derivative; a measure of instantaneous rate of change for any (even nonlinear) functions; tangent line to a curve at a point
Real World Situation (R)	Static, physical or dynamic, functional situation (e.g., wheelchair ramp, distance versus time)
Determining Property (D)	Property that determines if lines are parallel or perpendicular; property can determine a line if a point on the line is also given
Behavior Indicator (B)	Property that indicates increasing/decreasing/horizontal trends of line or amount of increase or decrease; if nonzero, indicates intersection with x-axis
Linear Constant (L)	Constant property independent of representation; unaffected by translation of a line; reference to what makes a line "straight" or the "straightness" of a line

These 11 conceptualizations point to the diversity and complexity of slope in the mathematics curriculum. While past research has investigated the prevalence of each of these conceptualizations among individuals who have completed their K-12 education (Nagle, Moore-Russo, Viglietti, & Martin, *in press*; Moore-Russo, Conner, & Rugg, 2011; Mudaly & Moore-Russo, 2011; Stanton & Moore-Russo, 2012), the focus so far has been an isolated description of the conceptualizations without consideration of how they may fit together to form an individual's network of slope conceptions. In the theoretical contribution provided in this article, we reanalyze the 11 conceptualizations in light of one another and research on understanding. In particular, we combine research on procedural versus conceptual understanding with research on visual and analytic interpretations of slope to draw new connections between the conceptualizations. A review of the relevant research is followed by an application of these ideas to the 11 conceptualizations, resulting in a description of a slope conceptualization network with five underlying components.

Procedural versus Conceptual Knowledge

Since Skemp (1967) first described two distinct bodies of mathematics, termed relational and instrumental, considerable research has focused on what is better known now as procedural versus conceptual understanding. Hiebert and Lefevre (1989) describe procedural knowledge as knowing "how to" and conceptual knowledge as knowing "why". More recently, procedural knowledge was described as knowledge of rules and processes linked to specific problems while conceptual knowledge was said to involve a more flexible understanding of governing rules that can be transferred to various problems and situations (Rittle-Johnson, Siegler, & Alibali, 2001). While researchers have taken different stances on the relationship between these two types of understanding and the appropriate developmental sequence (e.g., conceptual then procedural, procedural then conceptual, or simultaneous development of the two), there seems to be consensus that both play an important role in a student's knowledge of mathematics (Geary, 1994; Halford, 1993; Hiebert & Wearne, 1996; Rittle-Johnson & Alibali, 1999).

Findings suggest many students hold only procedural knowledge of the algorithms and interpretations of slope in specific situations, such as rote application of "change in y over change in x" or "rise over run" to find the slope of a line given two points through which it passes. Although students might be able to identify pairs of lines as parallel or perpendicular based on the coefficients of the x-terms in algebraic representations of the form $y=mx+b$, they often do not understand how to use slope to determine the behavior of a linear graph or the

relationship between nonparallel and nonperpendicular lines. Reiken (2008) described students' understanding of slope as procedural and linked this procedural knowledge to students' view of slope as a number in five contexts: number from formula, number from counting, number in front of x, number as relator, and number as rate of change. While the number as relator (relating x and y) and number as rate of change conceptualizations may suggest some conceptual understanding, the remaining interpretations of slope as a number are procedural (Reiken 2008).

Visual Versus Analytic Interpretations of Slope

Students' strategies for solving mathematical tasks are also of interest to researchers. Lowrie and Clements (2001) distinguished between visual and non-visual (verbal/analytic) approaches to mathematical tasks, and described students' transition to more non-visual emphases throughout a school year. Zaslavsky, Sela, and Leron (2002) described analytic and visual interpretations of slope. Analytic interpretations emphasize slope as a "property of the function" that "does not depend on the function's representation" (Zaslavsky et al., p. 122). Notions of derivatives, difference quotients, and the coefficient m were described as analytic interpretations due to their functional emphasis. By contrast, the visual perspective treats slope as a "property of a line" (p. 122), i.e., as the angle formed between the horizontal axis and the line or as the ratio of the vertical change to the horizontal change (Zaslavsky et al., 2002). This study showed the cognitive conflict that arose in the minds of mathematicians who were asked to find the slope of two lines, one of which was represented on a non-homogenous coordinate system (different scales on the horizontal and vertical axes). Using an analytic perspective, the slope of the lines were equal ($m=1$). However, a visual perspective focused participants' attention on the angles of inclination formed between the lines and the horizontal axes; angles that appeared to have different measures under the two scaling systems. As a result, the authors argue that an analytic perspective is superior to a visual perspective since it holds under both homogenous and non-homogenous coordinate systems.

Combining Theories: A Slope Conceptualization Network

The theory proposed in this study builds on: (a) previous research related to slope; (b) general theories related to understanding; and (c) data collected for slope research studies conducted from 2007 to 2012. The data referenced included mathematics standards documents as well as observations of and written artifacts collected from a variety of individuals who should have already experienced K-12 instruction in slope, including: practicing middle and high school mathematics teachers; high school graduates enrolled in introductory college calculus courses; college graduates with Bachelor's degrees in mathematics, engineering, childhood education, or mathematics education enrolled in graduate teacher education courses; individuals with Master's degrees in either mathematics or mathematics education enrolled in mathematics education doctoral programs; and post-secondary mathematics instructors. These data were used to revisit examples of the 11 conceptualizations in light of: (a) procedural versus conceptual understanding and (b) analytic versus visual interpretations. The original 11 conceptualizations were scrutinized, with particular attention to their connections. The results of this analysis led to the collapsing of some previously distinct conceptualizations. As a result, the researchers constructed a 2x2 matrix indicating procedural and analytic, procedural and visual, conceptual and analytic, and conceptual and visual dimensions for each identified component of slope. A more detailed description of the identified slope components is in the next section.

Components of the Slope Conceptualization Network

When distinctions were made in terms of procedural versus conceptual understanding and visual versus analytic interpretations, connections and relationships between the previously distinct conceptualizations became apparent. Under this lens, five key slope components emerged: (1) constant ratio, (2) determining property, (3) behavior indicator, (4) trigonometry, and (5) calculus. In addition, real world situations were determined to be possible with each of the five components. A description of the four dimensions associated with each slope component, including relationships to the original 11 conceptualizations, follows.

Constant ratio. Table 2 outlines the constant ratio component. Previously distinct algebraic ratio and geometric ratio conceptualizations merge when visual and analytic interpretations of ratio are considered. A student with a visual approach to slope as a ratio might interpret slope as the ratio of rise over run or the vertical change divided by the horizontal change of two points on a linear graph. A more conceptual, but still visual interpretation would include validating the ratio using the notion of triangle similarity to recognize the constancy of the rise/run ratio. An analytic approach to slope as a ratio involves defining slope as the change in y values over the change in x values. At the conceptual level, the analytic ratio component recognizes slope as a constant rate of change between two covarying quantities. In addition to combining algebraic and geometric ratios via analytic versus visual distinctions, two other important conceptualizations were absorbed into the constant ratio component by consideration of the conceptual versus procedural approach. Notice that the conceptual level of slope as a ratio includes recognizing slope as a constant ratio for linear functions. In other words, conceptual understanding of slope as a ratio requires recognition that this ratio is independent of the particular location on the graph or input/output pairs that are chosen. Thus, the linear constant conceptualization (see Table 1) was absorbed at the conceptual level of the new constant ratio component. Likewise, a conceptual and analytic understanding of slope as a constant ratio involves recognizing slope as a constant rate of change of two covarying quantities, previously known as functional property. Four previously distinct conceptualizations fit under one umbrella when the procedural versus conceptual emphasis and visual versus analytic interpretations are considered. Note that previous conceptualizations are listed in Table 2 using square brackets.

Table 2: Examples of Slope as a Constant Ratio

	Procedural Emphasis	Conceptual Emphasis
Visual Approach	$R_{v,p}$: rise/run or vertical change/horizontal change [previously geometry ratio]	$R_{v,c}$: similarity of slope triangles yields a constant ratio of rise/run regardless of the position on the graph [previously linear constant]
Analytic Approach	$R_{a,p}$: change in y over change in x; (y_2-y_1/x_2-x_1) [previously algebraic ratio]	$R_{a,c}$: constant rate of change between two covarying quantities; an equivalence class of ratios and hence a function [previously functional property]

Behavior indicator. The behavior indicator component, outlined in Table 3, relates slope to the increasing, decreasing, or constant behavior of a linear graph (visual approach) or function (analytic approach). Under the analytic approach, students first recognize slope as m in the equation of a linear function and then interpret the sign of m to indicate the function behavior. This recognition of slope as m was previously classified as the parametric coefficient

conceptualization. To move to conceptual understanding of slope as a behavior indicator, students must first be able to interpret slope as a ratio. For a visual approach, students can link the increasing or decreasing behavior of a linear graph to positive or negative values of rise and run, which then leads to a positive or negative ratio of rise/run. In the analytic approach, students use the definition of an increasing or decreasing function to determine the sign of the change in y/change in x ratio. The constant ratio component appears foundational to a conceptual understanding of slope as a behavior indicator.

Table 3: Examples of Slope as a Behavior Indicator

	Procedural Emphasis	Conceptual Emphasis
Visual Approach	$B_{v,p}$: increasing lines have positive slope; decreasing lines have negative slope; horizontal lines have zero slope	$B_{v,c}$: positive rise corresponds to positive run for an increasing line, yielding a positive slope. For a decreasing line, a negative rise corresponds to a positive run, yielding a negative slope. A horizontal line has zero rise for any run, yielding a zero slope.
Analytic Approach	$B_{a,p}$: value of m in the equation for a linear function (e.g., in $y=mx+b$) indicates whether f is an increasing ($m>0$), decreasing ($m<0$), or constant ($m=0$) linear function [previously parametric coefficient]	$B_{a,c}$: application of the definition of increasing/decreasing/constant functions to explain positive/negative/zero slope, respectively (e.g., f is increasing means that $f(x_1)<f(x_2)$ if $x_1<x_2$, so $[f(x_2)-f(x_1)]/(x_2-x_1)>0$)

Determining property. Table 4 shows that the determining property component involves recognizing that slope can: (a) determine the relationship between lines (e.g., parallel) and (b) indicate that a point and slope determine a unique line. While a procedural emphasis requires that students recognize these relationships, a conceptual emphasis includes understanding the underlying features of the linear graphs or functions that yield these relationships.

Table 4: Examples of Slope as a Determining Property

	Procedural Emphasis	Conceptual Emphasis
Visual Approach	$D_{v,p}$: parallel, coplanar lines have the same slope; perpendicular, coplanar lines (lines that intersect at right angles) have negative reciprocal slopes; slope and a point determine a unique line	$D_{v,c}$: parallel lines have the same vertical change for a set horizontal change (otherwise they would intersect); may be seen in terms of congruent slope triangles
Analytic Approach	$D_{a,p}$: ratio (y_2-y_1/x_2-x_1) is equivalent for parallel lines and results in negative reciprocals for perpendicular lines; slope and a point determine a unique linear equation	$D_{a,c}$: parallel lines have equivalent differences in y values for a set difference in x values, yielding equivalent slope ratios

Trigonometry. The trigonometric component of slope is described in Table 5. Once again, distinguishing between visual and analytic interpretations of slope allowed previously distinct slope conceptualizations to be combined under one category. A focus on steepness, which was previously identified as a physical property conceptualization, can be seen as the manifestation of a visual approach of the trigonometric component. Procedurally, the focus is on determining the angle of inclination, whereas a conceptual focus includes viewing the ratio of rise over run in terms of the opposite and adjacent sides of the right triangle formed by the line and the horizontal. The analytic approach to slope in terms of the trigonometry component involves the procedure of calculating $\tan\theta$. At the conceptual level, a student can relate the angle of inclination to the ratio (y_2-y_1/x_2-x_1), also known as $\tan\theta$.

Table 5: Examples of Slope as a Trigonometric Conception

	Procedural Emphasis	Conceptual Emphasis
Visual Approach	$T_{v,p}$: steepness of a line; slope as the angle of inclination of the line with a horizontal; as a line is rotated about a point, the slope changes [previously physical property]	$T_{v,c}$: the angle of inclination determines the rise/run; a steeper line has a greater rise per given run than a less steep line
Analytic Approach	$T_{a,p}$: slope is calculated as $\tan\theta$, where θ is the angle formed by the graph of the linear equation and an intersecting horizontal line	$T_{a,c}$: the angle of inclination determines the ratio of (y_2-y_1/x_2-x_1), which is equivalent to $\tan\theta$

Calculus. The calculus component, outlined in Table 6, involves interpreting slope of a function at a point. Visually, this is done via the slope of the tangent line to the point, with a conceptual emphasis on using secant line approximations to find the slope. Analytically, the slope is found via the derivative function (either via the limit definition or the various shortcuts for finding the derivative). A conceptual emphasis involves understanding the limit definition as describing the average rate of change over increasingly small intervals. Since the visual approach relies on first understanding slope of a line using the rise/run ratio and the analytic approach relies on first understanding slope of a linear function using the change in *y*/change in *x* ratio, the calculus component of slope appears to be closely linked to the ratio component.

Table 6: Examples of Slope in Calculus

	Procedural Emphasis	Conceptual Emphasis
Visual Approach	$C_{v,p}$: slope of a curve at a point is the slope of the tangent line to the curve at a given point	$C_{v,c}$: visual interpretation of secant lines approaching tangent line to understand slope of a curve at a point as the slope of the tangent line at the point
Analytic Approach	$C_{a,p}$: derivative f' is used to calculate slope of the function f at a particular point	$C_{a,c}$: interpreting $f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$ as the average rate of change over increasingly small intervals

Real world situations and parametric coefficient. One of the original 11 conceptualizations does not appear above. Real world situations can be incorporated with any one (or any combination) of the outlined slope components. For instance, references to the pitch of a roof would incorporate real world situations into the trigonometry component to determine the steepness of the roof (possibly using the angle formed between the roof line and the ceiling of house's interior), while references to speed as the rate of change between distance and time would incorporate real world situations with the constant ratio component to interpret speed as a ratio describing a relationship between covarying distance and time quantities. Table 7 outlines the components of the slope network as they relate to the original 11 conceptualizations.

Table 7: Revised Slope Network Components and Corresponding Conceptualizations

	Slope Component	Description	Underlying Slope Conceptualizations
Real World Situations (span each component)	Constant Ratio	Slope viewed as a ratio in visual (rise/run) or analytic (change in y over change in x) form; conceptual understanding extends to explain why linear behavior results in a constant ratio (including functional property for analytic representation)	Geometric Ratio Algebraic Ratio Functional Property Linear Constant
	Trigonometric Conception	Describes slope in terms of the angle of inclination of a line with a horizontal; conceptual understanding relates steepness to the determination of the tangent of the angle of inclination	Physical Property Trigonometric Conception
	Behavior Indicator	Relates slope to the increasing or decreasing behavior of a linear function or graph; links sign of the quantity m with the function or graph's behavior	Parametric Coefficient Behavior Indicator
	Calculus Conception	Limit; derivative; a measure of instantaneous rate of change for any (even nonlinear) functions; tangent line to a curve at a point	Calculus Conception
	Determining Property	Property that determines if lines are parallel or perpendicular; property can determine a line if a point on the line is also given	Determining Property

Discussion and Conclusion

The above analysis merges past research on students' understanding of slope to provide a more connected understanding of the components of an individual's network of slope concepts. In particular, the analysis suggests understanding slope can be viewed in light of five components, each with four subcomponents defined by the conceptual versus procedural understanding and analytic versus visual interpretations. While it has been argued in the past that students must build a particular type of understanding of slope (e.g., conceptual or analytic), the network interpretation of slope highlights the importance of procedural and conceptual understanding, as well as visual and analytic interpretations of slope. While Zaslavsky and colleagues (2002) suggested that a visual perspective of slope might be limiting, the network suggests that visual interpretations are an important part of an individual's network of slope components. We would argue that limitations stem from a lack of connections between the various components and subcomponents of slope. Not only should instruction build each of the five slope components, but explicit attention should be given to building connections between

the subcomponents. This means that students must have an opportunity to build and connect slope representations that are analytic, visual, procedural, and conceptual.

A wealth of research supports that slope is a critical topic and that students struggle to build a strong understanding of this important topic. We have contributed to the theory for how slope is understood by synthesizing past research to describe a network of slope components and subcomponents that together build a strong and diverse view of slope. This theory can strengthen future research on slope by tying together the important aspects of understanding slope under one model.

References

Barr, G. (1981). Student ideas on the concept of gradient. *Mathematics in School, 10,* 16-17.

Carlson, M., Jacobs, S., Coe, E., Larsen, S., & Hsu, E. (2002). Applying covariational reasoning while modeling dynamic events: A framework and a study. *Journal for Research in Mathematics Education, 33,* 352-378.

Carlson, M., Oehrtman, M., & Engelke, N. (2010). The precalculus concept assessment: A tool for assessing students' reasoning abilities and understandings. *Cognition and Instruction*, *28*, 113-145.

Confrey, J., & Smith, E. (1995). Splitting, covariation, and their role in the development of exponential functions. *Journal for Research in Mathematics Education, 26,* 66-86.

Geary, D. C. (1994). *Children's mathematical development: Research and practical applications.* Washington, DC: American Psychological Association.

Halford, G. S. (1993). *Children's understanding: The development of mental models.* Hillsdale, NJ; Erlbaum.

Hattikudur, S., Prather, R.W., Asquith, P., Knuth, E., Nathan, M. J., & Alibali, M. W. (2011). Constructing graphical representations: Middle schoolers' developing knowledge about slope and intercept. *School Science and Mathematics*, *112*(4), 230-240.

Hiebert, J., & Wearne, D. (1996). Instruction, understanding and skill in multidigit addition and instruction. *Cognition and Instruction, 14,* 251-283.

Lobato, J., & Siebert, D. (2002). Quantitative reasoning in a reconceived view of transfer. *Journal of Mathematical Behavior*, *21*, 87-116.

Lobato, J., & Thanheiser, E. (2002). Developing understanding of ratio-as-measure as a foundation of slope. In B Litwiller and G. Bright (Eds.), *Making sense of fractions, ratios, and proportions*, pp. 162-175. Reston, VA: The National Council of Teachers of Mathematics.

Lowrie, T., & Clements, M. A. (2001). Visual and nonvisual processes in grade 6 students' matheamtical problem solving. *Journal of Research in Childhood Education*, *16*(1), 77-93.

Moore-Russo, D., Conner, A., & Rugg, K. I. (2011). Can slope be negative in 3 space? Studying concept image of slope through collective definition construction. *Educational Studies in Mathematics, 76*(1), 3-21.

Mudaly, V. & Moore-Russo, D. (2011). South African teachers' conceptualisations of gradient: A study of historically disadvantaged teachers in an Advanced Certificate in Education programme. *Pythagoras, 32*(1), 27-33.

Nagle, C., Moore-Russo, D., Viglietti, J., & Martin, K. (2013). Calculus students' and instructors' conceptualizations of slope: A comparison across academic levels. *International Journal of Science and Mathematics Education*. Advance online publication. doi: 10.1007/s10763-013-9411-2

Noble, T., Nemirovsky, R, Wright, T., & Tierney, C. (2001). Experiencing change: The mathematics of change in multiple environments. *Journal for Research in Mathematics Education*, *32*, 85-108.

Orton, A. (1984). Understanding rate of change. *Mathematics in School, 5,* 23-26.

Reiken, J. (2008). *Coming to understand slope and the Cartesian connection: An investigation of student thinking* (Doctoral dissertation). Retrieved from Proquest Dissertations & Theses. (3347606)

Rittle-Johnson, B., & Alibali, M. W. (1999). Conceptual and procedural knowledge of mathematics: Does one lead to the other? *Journal of Educational Psychology*, *91*, 175-189.

Rittle-Johnson, B., Siegler, R., Alibali, M.W. (2001). Developing conceptual understanding and procedural skill in mathematics: An iterative process. *Journal of Education Psychology*, *93*, 346-362.

Stanton, M., & Moore-Russo, D. (2012). Conceptualizations of slope: A look at state standards. *School Science and Mathematics, 112*(5), 270-277.

Stump, S. (1999). Secondary mathematics teachers' knowledge of slope. *Mathematics Education Research Journal*, *11*, 124-144.

Stump, S. (2001a). Developing preservice teachers' pedagogical content knowledge of slope. *Journal of Mathematical Behavior, 20*, 207-227.

Stump, S. (2001b). High school precalculus students' understanding of slope as measure. *School Science and Mathematics*, *101*, 81-89.

Teuscher, D., & Reys, R. (2010). Slope, rate of change, and steepness: Do students understand these concepts? *Mathematics Teacher*, *103*, 519-524.

Thompson, P. W. (1994). The development of the concept of speed and its relationship to concepts of rate. In G. Harel & J. Confrey (Eds.), *The Development of Multiplicative Reasoning in the Learning of Mathematics*, pp. 179-234. Albany, NY: SUNY Press.

Zaslavsky, O., Sela, H., & Leron, U. (2002). Being sloppy about slope: The effect of changing the scale. *Educational Studies in Mathematics*, *49*, 119-140.